

# A lossy transmission line as a quantum open system in the standard quantum limit

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We systematically investigate how to quantize a transmission line resonator (TLR) in a mesoscopic electrical circuit in the presence of the resistance and the conductance of the dielectric media. Developed from the quantum bath based effective Hamiltonian method for single mode harmonic oscillator, the approach we presented in this article is a microscopic theory integrating quantum fluctuation-dissipation relation. To qualitatively check the condition under which the TLR can behave as a quantum object we study the classical-quantum boundary characterized by the standard quantum limit.

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An ideal one-dimensional transmission line resonator (TLR) can be described by a classical wave equation and thus can be quantized [1] as usual by following a standard procedure - the canonical quantization approach [2]. Such a quantization formalism is not treated as a serious issue in the usual realities. Most recently, however, the situation has been changed radically due to the rapid progresses in solid state based quantum computing (QC). In one of such QC schemes a one-dimensional TLR is used to coherently couple one or more Josephson junction (JJ) qubits [3–6]. In order to create controllable quantum entanglements among these JJ qubits, the TLR has to work in a quantum manner as a quantum data bus linking these qubits. Otherwise the TLR can not induce an effective inter-qubit interaction. One can imagine this in the conventional cavity QED: the classical cavity mode in the strong field limit do not induce inter-atom interactions to form quantum entanglement of the qubits.

The above arguments show that the validity of the TLR based quantum computing strongly depends on whether the mode of TLR is truly quantized, that is, has some observable quantum effects. In this sense, the way to quantize the modes of the TLR and the corresponding quantization condition become fundamentally important for applications of TLR in quantum information processing. In this article we devote to answer this question in a more realistic situation taking into consideration of leakage. Furthermore, we will try to find out what characterize the boundary between the quantum and classical regime for the lossy TLR by examining the so-called standard quantum limit (SQL) [7,8], which, as a consequence of the Heisenberg uncertainty principle, is usually referred to as the fundamental limit of the precision of repeated position measurements. In the modern technology based quantum measurement it was recognized that if one can reach the SQL in the experiments, the quantum behavior is observable even for macroscopic objects. Most recently LaHaye *et. al.* [9] described such an experiment with the goal to test SQL on a vibrating nano-mechanical beam that is about one-hundredth of a millimeter. This excites our interest on the similar prob-

lem about the actual boundary between the classical and quantum regime for the TLR.

To start with we consider the quantization of the modes of the one-dimensional lossy TLR. In our model the lossy TLR is treated as an open system interacting with a bath, a thermal environment. It may be a background electromagnetic field interacting with the transmitted charge in the TLR, or the classical lead connected to the TLR, or other damping mechanism. Mathematically the lossy TLR can be well depicted by a wave equation with leakage and its Fourier components obey the typical dissipation equation. In this sense, the idea and methods developed in our previous works [10–12] on quantum dissipative system can be applied to the present discussion. We will show that the interaction with the bath leads to an explicit description for the TLR in terms of the generalized Caldirola-Kani (CK) effective Hamiltonian. We also examine how the quantum and thermal fluctuations of the environment contribute to the uncertainty of the canonical variables of the lossy TLR.

To be universal we firstly revisit the derivation of the classical wave function for the lossy TLR in high dimensional case. The model is depicted by four lumped parameters, the distributed resistance  $r$ , the distributed conductance  $g$  of the dielectric media, the distributed inductance  $l$  and distributed capacitance  $c$  per unit length. Let  $V = V(\mathbf{x}, t)$  and  $\mathbf{I} = \mathbf{I}(\mathbf{x}, t)$  be the distributions of the voltage and the current vector respectively. With the conservation of the current, the Kirchhoff's voltage law leads to the equations of motion [13]

$$\nabla V = -r\mathbf{I} - l \frac{\partial \mathbf{I}}{\partial t}, \quad \nabla \cdot \mathbf{I} = -gV - c \frac{\partial V}{\partial t} \quad (1)$$

Eliminating the current vector in the above two equations we obtain the high dimensional lossy wave equation for the voltage [14].

$$\nabla^2 V = rgV + (rc + gl) \frac{\partial V}{\partial t} + \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} \quad (2)$$

where  $v = \sqrt{\frac{1}{lc}}$  is the velocity of propagation or phase speed. For an isolated conductor there does not exist the current along the norm direction of the surface

of conductor and one thus have the boundary condition  $\mathbf{I}_{\mathbf{n}|boundary} \equiv \mathbf{I} \cdot \mathbf{n}|_{boundary} = 0$  where  $\mathbf{n}$  is the direction of the norm of the boundary surface. Combining with the equations of motion, Eq.(1) leads to the boundary condition  $\mathbf{n} \cdot \nabla V|_{boundary} = \mathbf{n} \cdot (-r\mathbf{I} - l\frac{\partial \mathbf{I}}{\partial t}) = 0$  for the voltage equation. To be specific in one dimensional case, we have the boundary conditions for the current  $I(x=0,t) = I(x=L,t) = 0$  of an isolated 1-d TLR of length  $L$ . The corresponding boundary conditions for voltage equation is  $V_x(x=0,t) = V_x(x=L,t) = 0$ . In the following discussion we will focus on the quantization of the 1-d lossy TLR with such boundary conditions.

A canonical quantization scheme for an ideal 1-d lossless TLR can be found in Louisell's monograph with an implied Hamiltonian [1]. But for the lossy TLR as an open system the energy does not conserve and thus there does not exist a time-independent Hamiltonian a priori. Hence it is necessary to develop a quantization scheme only based on the classical equation of motion. In the very original paper of quantum mechanics by Heisenberg, however, without knowing the Hamiltonian or Lagrangian beforehand the classical equation of motion is sufficient to solve the quantized energy levels of linear oscillator. The modern version of this idea was presented in the famous textbook by Landau and Lifshitz [15]. Following this method we even quantized the quantum dissipation system [10–12] to re-deduce the so-called CK effective Hamiltonian [16–18] directly from the classical equation of motion for the open system with the linear coupling to the Ohmic bath [19].

Now we quantize the modes of the lossy TLR by developing our previous approach mentioned above. We can start from the lossy voltage equation

$$\frac{1}{v^2} \frac{\partial^2}{\partial t^2} V + (lg + rc) \frac{\partial}{\partial t} V + rgV - \frac{\partial^2 V}{\partial x^2} = 0 \quad (3)$$

with boundary condition  $V_x(0,t) = V_x(L,t) = 0$ . The time-dependent Fourier's components  $V_n(t)$  of  $V(x,t)$  corresponding the normal mode  $\sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}$  just satisfies the dissipation equation

$$\frac{d^2}{dt^2} V_n(t) + \gamma \frac{d}{dt} V_n(t) + \omega_n^2 V_n(t) = 0 \quad (4)$$

where the dissipation rate and the effective frequency are  $\gamma = v^2(gl + rc)$  and  $\omega_n^2 = v^2[rg + (\frac{n\pi}{L})^2]$  respectively.

To find out the canonical commutation relation for the dynamical variables  $V_n(t)$  and  $\frac{d}{dt} V_n(t)$  we calculate the time evolution of their commutator  $B(t) = [V_n, \frac{d}{dt} V_n(t)]$  using the equation of motion (4), obtaining the close equation  $\frac{d}{dt} B = -\gamma B$ . That means  $B \propto e^{-\gamma t}$ , or

$$[V_n, \frac{d}{dt} V_n(t)] = i\hbar e^{-\gamma t} / M_n. \quad (5)$$

Here we choose the effective mass  $M_n = \frac{c}{\omega_n^2}$  so that we can get a correct expression for the energy of TLR. Therefore, we can define the canonical momentum operators

$$P_n(t) = M_n e^{\gamma t} \frac{d}{dt} V_n(t) \quad (6)$$

to realize the conventional canonical commutation relation  $[V_n, P_m] = i\hbar \delta_{nm}$ .

In terms of the annihilation and creation operators  $a_n(t)$  and  $a_n^\dagger(t)$  defined by

$$V_n(t) = \sqrt{\frac{\hbar \omega_n}{2c}} (a_n + a_n^\dagger),$$

$$P_n(t) = -i\sqrt{\frac{\hbar c}{2\omega_n}} (a_n - a_n^\dagger) \quad (7)$$

the C-K Hamiltonian for the 1-D lossy TLR is obtained as  $\mathcal{H} = \sum_n H_n$ , where

$$H_n \equiv \hbar \omega_n \{ [a_n^{\dagger 2} \sinh(\gamma t) + a_n^\dagger a_n \cosh(\gamma t)] + h.c. \}. \quad (8)$$

Here we have ignored a time-dependent c-number  $\hbar \omega_n \cosh(\gamma t)$

The above effective Hamiltonian  $\mathcal{H}$  can force the TLR to evolve into the multi-mode squeezed state when the TLR is initially prepared in a multi-mode coherent state  $|\alpha\rangle \equiv |\alpha_1, \alpha_2, \dots, \alpha_n, \dots\rangle$  where  $|\alpha\rangle_n$  denotes the  $n$ -th mode. This conclusion can be proved by rewriting  $H_n$  as

$$H = \hbar \omega'_n A_n^\dagger A_n = S_n^\dagger(t) \hbar \omega_n a_n^\dagger a_n S_n(t) \quad (9)$$

Here,  $\{A_n\}$  is a new set of bosonic operator defined as the unitary transformation of  $a_n$   $A_n = S_n^\dagger(t) a_n S_n(t)$  by the squeezing operators [20]

$$S_n(t) = \exp \left[ \frac{1}{4} \gamma t (a_n^{\dagger 2} - a_n^2) \right] \quad (10)$$

We can also write down the explicit expressions  $A_n = \xi a_n - \eta a_n^\dagger$  with  $\xi = \cosh(\gamma t/2)$ ,  $\eta = -\sinh(\gamma t/2)$ . Then we obtain the evolution operator

$$U_n(t) = S_n^\dagger(t) \exp[-i\omega_n a_n^\dagger a_n t] S_n(0), \quad (11)$$

which gives the the final state  $|\Psi_n(t)\rangle = U_n(t) |\alpha\rangle_n = |\alpha e^{-i\omega_n t}, \xi, -\eta\rangle_n$  as a squeezed state defined as the eigenstate of  $A_n$  with eigenvalue  $\alpha e^{-i\omega_n t}$ .

Studying the above results carefully, it seems that the above results are not totally convincing due to the violation of the uncertainty principle about coordinate  $V_n(t)$  and momentum  $P_n(t)$  because the above arguments are too phenomenological and the source of dissipation is not considered microscopically. Thus the Brownian motion can not be analyzed in the frame of the effective CK Hamiltonian formalism. What's more, if we would use the above CK Hamiltonian without restriction, some ridiculous conclusions are to be reached. Fortunately, we can solve this problem by demonstrating the derivation of the phenomenological CK Hamiltonian, starting with

the conventional system-plus-reservoir approach. Suppose the environment is a bath of many harmonic oscillators linearly coupled to the open system. The bath and system constitute a conservative composite system and thus its quantization is rather straightforward. As shown in our previous works, the total wave function is partially factorized with respect to system and bath when the Brownian fluctuation can be ignored under certain conditions [10]. the factorized part of the system is just the CK effective wave function governed by the CK Hamiltonian. With these recalls we now consider the damping mode equation

$$\ddot{V}_n(t) + \gamma \dot{V}_n(t) + \omega_n^2 V_n(t) = f_n(t) \quad (12)$$

where the parameter  $\gamma$  and  $\omega_n$  are the same as before and

$$f_n(t) = - \sum_j c_{jn} (x_{nj0} \cos \omega_{nj} t + \dot{x}_{nj0} \frac{\sin \omega_{nj} t}{\omega_{nj}}) \quad (13)$$

is a Brownian driving force. Here,  $x_{j0}$  ( $\dot{x}_{j0}$ ) are the initial values of the canonical coordinate (its first derivative with respect to  $t$ ) of the harmonic oscillators of the bath that coupled to the transmission line,  $\omega_{nj}$  is the frequency of the  $j$ -th oscillator of the independent reservoir coupling to the  $n$ -th mode of QTL with the corresponding coupling constant  $c_{jn}$ .

The solution  $V_n(t) \equiv Q_n(t) + \sum_j \xi_{nj}(t)$  of the above motion equation (12) can be solved as a direct sum of the dissipative motion

$$Q_n(t) = a_1(t) V_{n0} + a_2(t) \dot{V}_{n0} \quad (14)$$

and the quantum fluctuation of the reservoir.

$$\xi_{nj}(t) = \sum_j b_{nj1}(t) x_{nj0} + b_{nj2}(t) \dot{x}_{nj0} \quad (15)$$

where the coefficients  $a_1(t) = \frac{1}{\nu - \mu} (\nu e^{-\mu t} - \mu e^{-\nu t})$ ,  $a_2(t) = \frac{1}{\nu - \mu} (e^{-\mu t} - e^{-\nu t})$  with  $\mu = \frac{\gamma}{2} + i\omega'_n$ ,  $\nu = \frac{\gamma}{2} - i\omega'_n$ ,  $\omega'_n = \sqrt{\omega_n^2 - \frac{\gamma^2}{4}}$  and we ignore the lengthy and too tedious expressions of  $b_{nj1}(t)$  and  $b_{nj2}(t)$ .

To revisit the quantum effect of QTL, we need to consider the standard quantum limit (SQL). If the measurement that probes it can reach the accuracy of the SQL, then the quantum effect is observable. Obviously the standard quantum limit of  $V_n$  is contributed by the measurement of QTL and the bath fluctuation

$$(\Delta V_n^{SQL})^2 = (\Delta Q_n)^2 + \sigma_n^2 \quad (16)$$

where  $\Delta Q_n$  is the variation of  $Q_n(t)$  and

$$\sigma_n^2 = \sum_j (\Delta \xi_{nj})^2 \quad (17)$$

is just the width of Brownian motion. The first part of the above equation is just an average over the pure quantum state of the TLR. Thus we can use the uncertainty principle to determine the value of SQL. According to the commutation relation Eq.(5), the uncertainty relation means

$$\Delta \dot{V}_n \geq \frac{\hbar \exp(-\gamma t)}{2M_n \Delta V_n} \quad (18)$$

Then

$$(\Delta Q_n)^2 \geq \frac{2\hbar}{M_n \omega_n'^2} e^{-2\gamma t} [\gamma \sin^2 \omega'_n t + \omega'_n \sin 2\omega'_n t] \quad (19)$$

It is easy to see that as time  $t \rightarrow \infty$ , the standard quantum limit  $|\Delta Q(t)| \rightarrow 0$  (see Fig.1). If the quantum fluctuation caused by the bath fluctuation  $\xi_{nj}(t)$  is ignored inappropriately, the standard quantum limit is zero after a long time evolution! We will show as follows that the quantum fluctuation contributes a nonzero part as compensation.

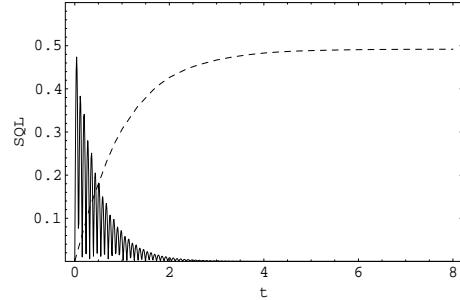


FIG. 1. The time evolution of the standard quantum limit of the system in low temperature limit. The solid line shows the evolution of the SQL of the TLR while the dashed line shows that of the environment. Here the time is scaled in the unit of  $\frac{1}{\gamma}$

We notice that the second part of Eq. (16) is the thermal average over the reservoir states and the fluctuation  $\sigma_n^2$  at temperature  $T$  is

$$\sigma_n^2(t) = \sum_j \coth \left( \frac{\hbar \omega_{nj}}{2k_B T} \right) \frac{\hbar [b_{nj1}^2(t) + \omega_{nj}^2 b_{nj2}^2(t)]}{2m_{nj} \omega_{nj}}. \quad (20)$$

where  $k_B$  is the Boltzmann constant and  $T$  is the temperature. This is just the width of the Brownian motion, which characterizes the extent of fluctuation around the damping path  $Q_n$ . In the classical limit, known as the "Ohmic friction" condition, the fluctuation can be evaluated as

$$\begin{aligned} \sigma_n^2(t) \approx & \frac{\hbar \gamma}{\pi M_n \omega'_n} \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) L(\omega) \\ & (1 - 2e^{-\frac{\gamma t}{2}} (\gamma \frac{\sin \omega'_n t}{2} + \omega'_n \cos \omega'_n t) \cos \omega t \\ & - 2\omega \sin \omega'_n t \sin \omega t) \end{aligned} \quad (21)$$

where  $L(\omega) = \omega/[(\omega_n^2 - \omega^2)^2 + \gamma^2\omega^2]$ .

As can be seen from Fig.1, in the low temperature limit  $\sigma_n^2(t)$  is zero initially, and then approaches its final equilibrium value in a time interval of the order of  $1/\gamma$ . The above results shows the important limit on the standard quantum limit of TLR caused by the environment fluctuation. It seemingly depends on the details of the reservoir, but they can be universally summed up to the observable quantities of the TLR in certain limit case. When the time is large enough, the total fluctuation of the whole system approaches (see Fig.2)

$$(\Delta V_n)_{SQL}^2(t \rightarrow \infty) = \frac{\hbar}{2\pi M_n \omega'_n} \left[ \frac{\pi}{2} + \arctan \left( \frac{\omega_n^2}{\gamma \omega'_n} \right) \right] \quad (22)$$

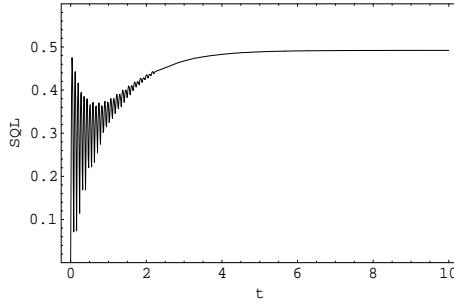


FIG. 2. The time evolution of the standard quantum limit of the whole system at low temperature limit. Again the time is scaled in the unit of  $\frac{1}{\gamma}$

If the damping rate  $\gamma$  is much smaller than the frequency of the oscillator  $\omega_n$ , this width happens to become the same as the width  $\hbar/(2M_n\omega_n)$  of the ground state of the mode, which is just the SQL of the quadrature amplitudes of the oscillator mode  $V_n(t)$ .

To conclude this article let us estimate the above result numerically according to the parameters given in the experimental proposal in ref [3,4]. The eigenfrequency of the TLR mode in resonant with the Josephson junction qubit is about 10 GHz while the dissipation rate is about  $6.25 \times 10^6$  Hz. Then

$$\sqrt{\frac{2}{L}} \Delta(V_n)_{SQL} \approx 0.2 \mu V \quad (23)$$

where  $L$  is the length of the TLR. Thus if the precision of the experiment can reach this limit, the quantum effect can be observed. We also note that our exploration in this article reveals the close relation between the SQL of the open system and the quantum fluctuation of the environment: starting from the initial semi-classical state, each mode of TLR experiences a damping squeezing (see Eqs.(9-11)) before it reaches the SQL. Once near the SQL the quantum fluctuation takes place to play as a quantum noise against the infinite squeezing by quantum dissipation.

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[13] To examine it in the two-dimensional case we consider the Green's theorem

$$\oint \mathbf{I} \cdot d\mathbf{l} = \oint (\nabla \cdot \mathbf{I}) d\mathbf{s}$$

where  $d\mathbf{l} = (-dy, dx)$  along the direction of tangent. The close path integral in the left hand side represents the all charges going into the domain surrounded by the closed path, which is just the sum of all leakages within this domain. we can consider the integral in the bulk cell  $\Delta x \Delta y$ , leading to the Eq.(1).

[14] For the current vector, however, instead of a lossy wave equation the modified equation of motion of current vector is

$$\nabla(\nabla \cdot \mathbf{J}) = RG\mathbf{J} + (Rc + lG) \frac{\partial \mathbf{J}}{\partial t} + lc \frac{\partial^2 \mathbf{J}}{\partial t^2}$$

But in one dimension, it can be reduced to the generic lossy wave equation.

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